# Syntax

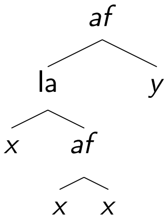
Let $V$ be a countable set of variables. The set of all well-formed terms, $\Lambda$, is defined inductively as follows :

- $V \subset \Lambda$
- $\lambda x.t \in \Lambda$ $\qquad \forall x \in V, \forall t \in \Lambda$
- $(t_1)t_2 \in \Lambda$ $\qquad \forall t_1, t_2 \in \Lambda$

# Syntax (cont'd)

The term $(\lambda x.(x)x)y$ has this syntactic structure :

$$af(la(x, af(x, x)), y)$$



# Variable substitution

- $x_{[x:=z]} \rightsquigarrow z$
- $y_{[x:=z]} \rightsquigarrow y$ si $y \neq x$
- $(M)N_{[x:=z]} \rightsquigarrow (M_{[x:=z]})N_{[x:=z]}$
- $\lambda x.M_{[x:=z]} \rightsquigarrow \lambda z.M_{[x:=z]}$
- $\lambda y.M_{[x:=z]} \rightsquigarrow \lambda y.M_{[x:=z]}$ if $x \neq y$

# Term substitution

- $x_{[x:=t]} \rightsquigarrow t$
- $y_{[x:=t]} \rightsquigarrow y$ si $y \neq x$
- $(M)N_{[x:=t]} \rightsquigarrow (M_{[x:=t]})N_{[x:=t]}$
- $\lambda y.M_{[x:=t]} \rightsquigarrow \lambda y.M_{[x:=t]}$ if $y$ is not free in $t$.

# $\alpha$ equivalence

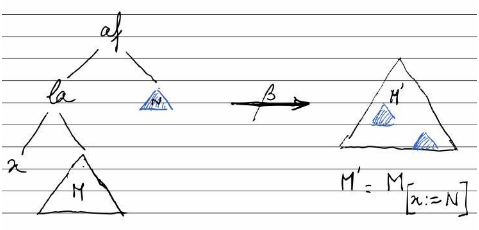
$$\lambda x.\varphi \equiv \lambda z.\varphi_{[x:=z]}$$

# Convention on variables

Let $M$ be a term, $x$ a variable. By convention, the occurrences of $x$ in $M$ are either all free or all bound.
It can be shown that every term constructed without respecting this convention is $\alpha$-equivalent to a term that respects the convention.

# $\beta$ equivalence

$$(\lambda x.M)N \equiv M_{[x:=N]}$$



# Combinators

A combinator is a closed $\lambda$-term (ie without free variable)

# Identity

$\mathsf{I} =_{\text{def}} \lambda x.x$

For any term $t$ : $(\mathsf{I})t \equiv t$

# Booleans

$\mathsf{T} =_{\text{def}} \lambda x.\lambda y.x$
$\mathsf{F} =_{\text{def}} \lambda x.\lambda y.y$

This encoding allows to encode an if-then-else function :

`if` $P$ `then` $Q$ `else` $R =_{\text{def}} ((P)Q)R$.

if $P$ is $\beta$-equivalent to T then $((P)Q)R$ will yield $Q$, while if $P$ is $\beta$-equivalent (or $\beta$-reduces) to F, the outcome will be $Q$.

# The IF combinator

$$\text{IF} =_{\text{def}} \lambda b.\lambda t.\lambda f.((b)t)f$$

$$\text{NOT} =_{\text{def}} \lambda u.((u)\text{F})\text{T}$$
$$\text{AND} =_{\text{def}} \lambda u.\lambda v.((u)v)\text{F}$$
$$\text{OR} =_{\text{def}} \lambda u.\lambda v.((u)\text{T})v$$

# Church numerals

$0 =_{\mathrm{def}} \lambda f.\lambda x.x$
$1 =_{\mathrm{def}} \lambda f.\lambda x.(f)x$
$\mathsf{n} =_{\mathrm{def}} \lambda f.\lambda x.(f)(f)\ldots(f)x$
with n times $f$

$\mathsf{Succ} =_{\mathrm{def}} \lambda n.\lambda f.\lambda x.(f)((n)f)x$
$+ \equiv \lambda m.\lambda n.\lambda f.\lambda x.((m)f)((n)f)x$
$* \equiv \lambda m.\lambda n.\lambda f.(m)(n)f$