Ex. 1

Translate as precisely as possible the following sentences into predicate logic. Explain the interpretation of non logical constants when it is not obvious. In case of ambiguity, propose as many formulae as necessary.

(1)    a.    Every politician is rejected by some voters.
       b.    Alex is upset as soon as everyone is noisy.
       c.    A child is confident only if no adult lies to him.
       d.    The moon has exactly two satellites.
       e.    When a problem is solved by all students, it should be suppressed.
       f.    All the newspapers which don't have readers will disappear if they don't find a buyer.
       g.    Either everyone takes a drink, or no one does.
       h.    Someone who refuses that everybody loves her should consult a doctor.

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(1a)  *Every politician is rejected by some voters.*
      We get here the usual ambiguity between a congruent (or *in situ*) reading (2a) or (2b) and an inverse scope reading (2c) or (2d).

      (2)    a.    $\forall x\,(Px \rightarrow \exists y\,(Vy \wedge Ryx))$
             b.    $\forall x\,\exists y\,(Px \rightarrow (Vy \wedge Ryx))$
             c.    $\exists y\,(Vy \wedge \forall x\,(Px \rightarrow Ryx))$
             d.    $\exists y\,\forall x\,(Vy \wedge (Px \rightarrow Ryx))$

(1b)  *Alex is upset as soon as everyone is noisy.*
      The connective *as soon as* is naturally considered as expressing a conditional. What makes Alex upset is the fact that everyone is noisy, which is expressed by the conditional structure in (3a). Note that while the formula in (3b) (or (3c) which is equivalent) looks very similar, it is in fact expressing something different: the fact that anyone being noisy is enough to make Alex upset. This of course entails that is everyone is noisy, Alex will be upset, but it is not equivalent.

      (3)    a.    $(\forall x\,(Px \rightarrow Nx)\ \rightarrow\ Ua)$
             b.    $\forall x\,((Px \rightarrow (Nx \rightarrow Ua))$
             c.    $\forall x\,((Px \wedge Nx) \rightarrow Ua)$

(1c)  *A child is confident only if no adult lies to him.*
      According to the most plausible reading, this sentence is a universal statement about the properties of children. We paraphrase this reading as (4a), formalized as (4b), which is equivalent to (4c) ($Tx$ means "$x$ is confident"). Note that *only if*, expressing a necessary condition, is translated into a material implication from the child being confident to no adult lying to them (and not the other way around). It is also possible to offer an existential reading for (1c), which could be paraphrased as (4d). This reading, which does not seem very plausible to me, would be formalized as (4e).

(4)    a.   *All children are confident only is no adult lies to them*
       b.   $\forall x \ (Cx \to (Tx \to \neg \exists y \ (Ay \wedge Lyx)))$
       c.   $\forall x \ ((Cx \wedge Tx) \to \neg \exists y \ (Ay \wedge Lyx))$
       d.   *There is a child who is confident only if no adult lies to them*
       e.   $\exists x \ (Cx \wedge (Tx \to \neg \exists y \ (Ay \wedge Lyx)))$

**(1d)**   *The moon has exactly two satellites.*
The moon can be represented as a constant $(m)$, considering it behaves like a proper noun. We then have to express that at least two individuals are in a satellite relation with it (first part of (5a)), and that no other individual stands in the same relation (second part of (5a)). It was also possible to express that there is (at leat) a moon (5b), or even that this moon is unique, in a neo-Russelian account (5c). (We assume that we can safely write $(A \wedge B \wedge C)$ for either $(A \wedge (B \wedge C))$ or $((A \wedge B) \wedge C)$, since they are logically equivalent.)

(5)    a.   $\exists x \exists y \ ((x \neq y \wedge Smx \wedge Smy) \wedge \forall z \ ((z \neq x \wedge z \neq y) \to \neg Smz))$
       b.   $\exists t \ (Mt \wedge \exists x \exists y \ ((x \neq y \wedge Stx \wedge Sty) \wedge \forall z \ ((z \neq x \wedge z \neq y) \to \neg Stz)))$
       c.   $\exists t \ (Mt \wedge \exists x \exists y \ ((x \neq y \wedge Stx \wedge Sty) \wedge \forall z \ ((z \neq x \wedge z \neq y) \to \neg Stz)) \wedge \neg \exists u \ (u \neq t \wedge Mu))$

**(1e)**   *When a problem is solved by all students, it should be suppressed.*
This is a typical donkey sentence, for which a strictly compositional translation yields an open formula (6a), but whose semantic content can be expressed with a non compositional representation (6b) ($Ex = x$ is a student; $Rx = x$ should be suppressed). Note that the property of being solved by all students, even though it is quantificational, does not interfere with the rest of the formula, which could be written as in (6c), with $Ax = x$ was solved by all students, or, if your prefer, $Ax = \lambda x.\forall y \ (Ey \to Sxy)$.

(6)    a.   $\big(\exists x \ (Px \wedge \forall y \ (Ey \to Sxy)) \to Rx\big)$
       b.   $\forall x \ ((Px \wedge \forall y \ (Ey \to Syx)) \to Rx)$
       c.   $\forall x \ ((Px \wedge Ax) \to Rx)$

**(1f)**   *All the newspapers which don't have readers will disappear if they don't find a buyer.*
The overall structure of this sentence could be summarized as in (7a), where $\mathcal{N}x = x$ is a newspaper, $\mathcal{R}x = x$ doesn't have readers, $\mathcal{B}x = x$ doesn't find a buyer, and $\mathcal{D}x = x$ disappears. To account for a possible large scope reading of the buyer (paraphrase in (7b)), the general structure is rather (7c), where $\mathcal{B}'xy = x$ does not find $y$. It is not too difficult to spell out $\mathcal{N}$, $\mathcal{R}$, $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{D}$: see (7d). Putting everything together yields (7e), which has quite a large number of variants, based on the equivalence between $(A \to (B \to C))$ and $((A \wedge B) \to C)$.

(7)    a.   $\forall x \ \Big((\mathcal{N}x \ \wedge \ \mathcal{R}x \ \wedge \ \mathcal{B}x) \ \to \ \mathcal{D}x\Big)$
       b.   *There is a buyer such that any newspaper not having readers will disappear if it does not find it*
       c.   $\exists y \ \Big(By \wedge \forall x \ \Big((\mathcal{N}x \ \wedge \ \mathcal{R}x \ \wedge \ \mathcal{B}'xy) \ \to \ \mathcal{D}xy\Big)\Big)$
       d.   $\mathcal{N}x = Nx,$
           $\mathcal{R}x = \neg \exists u \ (Pu \wedge Rux),$
           $\mathcal{B}x = \neg \exists v \ (Bv \wedge Fxv),$
           $\mathcal{B}'xy = \neg Fxy$
           $\mathcal{D}x = Dx$
       e.   $\forall x \ \Big(\big(Nx \wedge \neg \exists u \ (Pu \wedge Rux) \wedge \ \neg \exists v \ (Bv \wedge Fxv)\big) \to Dx\Big)$

(1g) *Either everyone takes a drink, or no one does.*
   The most natural reading is given in (8a), with a congruent interpretation of the
   quantifiers (universal having scope over existential). For each of the disjuncts,
   it also possible to inverse the scopes, which can be paraphrased as in (8b) and
   formalized as in (8c). But we could also come up with an interpretation where
   the indefinite *a drink* has scope even over the disjunction. The paraphrase is in
   (8d), the formala in (8e).

(8)    a.   $\left(\forall x\ (Px \to \exists y\ (Dy \wedge Txy))\ \vee\ \neg\exists x\ (Px \wedge \exists y\ (Dy \wedge Txy))\right)$
       b.   *Either there is one drink that everyone takes, or there is one drink*
            *that no one takes*
       c.   $\left(\exists y\ (Dy \wedge \forall x\ (Px \to Txy))\ \vee\ \exists y\ (Dy \wedge \neg\exists x\ (Px \wedge Txy))\right)$
       d.   *There is a drink which is such that either everyone takes it or no one*
            *does*
       e.   $\exists y\ \left(\forall x\ (Px \to Dxy) \vee \neg\exists x\ (Px \wedge Dxy)\right)$

(1h) *Someone who refuses that everybody loves her should consult a doctor.*
   Two aspects of this sentence are beyond our expressive power: (1) *refuse* + propo-
   sitional argument (second order); (2) modal and intensional reading. We can offer
   a formalization only by simplifying the sentence, removing second order pred-
   icates and modality. For instance, (9a) can be represented as (9b). Or Also:
   rejects anyone who loves her (send different)

(9)    a.   Everyone who is unhappy when everybody loves her will see a doctor.
       b.   $\forall x\ (Px \to ((\forall y\ (Py \to Lyx) \to Ux) \to \exists z\ (Dz \wedge Cxz)))$
       c.   Everyone who rejects people who loves them will see a doctor.
       d.   $\forall x\ (Px \to (\forall y\ (Py \wedge Lyx) \to Rxy) \to \exists z\ (Dz \wedge Cxz))$

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Ex. 2

Construct an analysis of *all but one* in terms of $\forall$, $\exists$, $=$, and propositional connectives,
and show how (10) would be analysed under your proposal.

(10)    All but one poet hated himself.

................................... Answer ...................................

- The general analysis for a sentence of the form "*All but one R S*" (R and S being
  two predicates) has to express that one individual of type R does not S, while all
  individual different from this individual indeed do S. Here is a general formula:
  $\exists x\ \left((Rx \wedge \neg Sx) \wedge \forall y\ ((Ry \wedge y \neq x) \to Sx)\right)$

- In (10), the predicate R corresponds to *poet*, while the predicate S corresponds to
  *hate oneself*. This gives:
  $\exists x\ \left((Px \wedge \neg Hxx) \wedge \forall y\ ((Py \wedge y \neq x) \to Hyy)\right)$

Ex. 3

In natural language (and also in science), discourse often has changing domains. Therefore it is interesting to study what happens to the truth of formulas in a model when the model undergoes some transformation. Let us call *persistent* a formula when its truth is not affected by enlarging the models with new objects. Which of the following formulae are generally persistent?

(11)    a.   $\exists x\ Px$
          b.   $\forall x\ Px$
          c.   $\exists x\forall y\ Rxy$
          d.   $\neg\forall x\forall y\ Rxy$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . Answer . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

The question focused on the truth of formulae (rather than on truth-values). So we take as a starting point a model which satisfies the formula (*i.e.*, a model in which the formula is true), and considerer the effects of enlarging the domain (*i.e.*, add new individuals).

(11a) $\exists x\ Px$

Adding individuals in the universe will not change the fact that at least one individual has the property $P$.                                    Persistent

(11b) $\forall x\ Px$

Since one individual could be added in the domain without this individual being in the set denoted by $P$, this formula may become false when the model is enlarged.
                                                                    Not persistent

(11c) $\exists x\forall y\ Rxy$

If this sentence is true, there are individuals (say, $\alpha_1, \alpha_2, \ldots \alpha_k$) such that it is true that all indivuals are in the $R$ relation with them (these individuals form the set of possible $x$'s that satisfy (11c)). We may add individuals such that they are in the $R$ relation with no individual of this set. In such a case, (11c) will become false.                                                                 Not persistent

(11d) $\neg\forall x\forall y\ Rxy$

Adding individuals won't change the fact that not all pairs of individuals are in the relation $R$.                                                     Persistent

If you insist on considering also the way enlarging the domain may have an impact when starting with a model **not** satisfying a given formula, you should note the two following points:

- We should not talk of *persistence* any more, we are dealing with a different property. Let's call *stability* the property of remaining false when the domain is enlarged. It is important to note that the two properties are independant. For instance, a formula like $Kj$ will be both persistent and stable, while (11a) is persistent and not stable (see below), and (11c) is neither persistent nor stable, *etc*.

- More importantly, asking the question of the stability of a formula $\varphi$ boils down to asking the question of the persistence of $\neg\varphi$. So after all we probably don't need an additional concept.

Since some of the students tried to answer the question of the stability of the initial formula, we give here the answers.

(11a) $\exists x\ Px$

A new individual may be added which would belong to the set denoted by $P$. Therefore the formula may become true.     Not stable ($\neg\exists x\ Px$ is not persistent)

(11b) $\forall x\ Px$

The falsity of this formula won't change if an individual is added, whether it has the property $P$ or not.                                  Stable ($\neg\forall x\ Px$ is persistent)

(11c) $\exists x\forall y\ Rxy$

A new individual may be added which stand in the $R$ relation with all other individuals, making the formula true.     Not stable ($\neg\exists x\forall y\ Rxy$ is not persistent)

(11d) $\neg\forall x\forall y\ Rxy$

If this sentence is false, then any pair of individuals stand in the $R$ relation. Adding a new individual that doesn't satisfy this property will make the sentence true.                                  Not stable ($\forall x\forall y\ Rxy$ is not persistent)