Compositionality & λ-calculus

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Overview

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Syntax

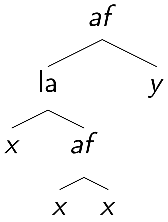
Let V be a countable set of variables. The set of all well-formed terms, Λ, is defined inductively as follows :

- $V \subset \Lambda$
- $\lambda x.t \in \Lambda$ $\quad \forall x \in V, \forall t \in \Lambda$
- $(t_1)t_2 \in \Lambda$ $\quad \forall t_1, t_2 \in \Lambda$

Syntax (cont'd)

The term $(\lambda x.(x)x)y$ has this syntactic structure :

$$af(la(x, af(x, x)), y)$$



Variable substitution

- $x_{[x:=z]} \rightsquigarrow z$
- $y_{[x:=z]} \rightsquigarrow y$ si $y \neq x$
- $(M)N_{[x:=z]} \rightsquigarrow (M_{[x:=z]})N_{[x:=z]}$
- $\lambda x.M_{[x:=z]} \rightsquigarrow \lambda z.M_{[x:=z]}$
- $\lambda y.M_{[x:=z]} \rightsquigarrow \lambda y.M_{[x:=z]}$ if $x \neq y$

Term substitution

- $x_{[x:=t]} \rightsquigarrow t$
- $y_{[x:=t]} \rightsquigarrow y$ si $y \neq x$
- $(M)N_{[x:=t]} \rightsquigarrow (M_{[x:=t]})N_{[x:=t]}$
- $\lambda y.M_{[x:=t]} \rightsquigarrow \lambda y.M_{[x:=t]}$ if y is not free in t.

α equivalence

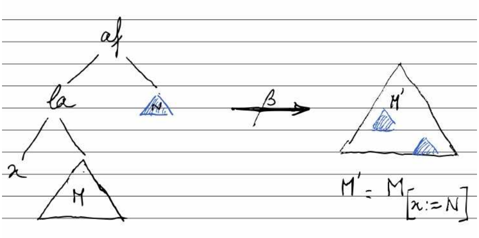
$$\lambda x.\varphi \equiv \lambda z.\varphi_{[x:=z]}$$

Convention on variables

Let M be a term, x a variable. By convention, the occurrences of x in M are either all free or all bound.
It can be shown that every term constructed without respecting this convention is α-equivalent to a term that respects the convention.

β equivalence

$$(\lambda x.M)N \equiv M_{[x:=N]}$$



Combinators

A combinator is a closed λ-term (ie without free variable)

Identity

$\mathsf{I} =_{\text{def}} \lambda x.x$

For any term t : $(\mathsf{I})t \equiv t$

Booleans

$\mathsf{T} =_{\text{def}} \lambda x.\lambda y.x$
$\mathsf{F} =_{\text{def}} \lambda x.\lambda y.y$

This encoding allows to encode an if-then-else function :

`if` P `then` Q `else` $R =_{\text{def}} ((P)Q)R$.

if P is β-equivalent to T then $((P)Q)R$ will yield Q, while if P is β-equivalent (or β-reduces) to F, the outcome will be Q.

The IF combinator

$$\text{IF} =_{\text{def}} \lambda b.\lambda t.\lambda f.((b)t)f$$

$$\text{NOT} =_{\text{def}} \lambda u.((u)\text{F})\text{T}$$
$$\text{AND} =_{\text{def}} \lambda u.\lambda v.((u)v)\text{F}$$
$$\text{OR} =_{\text{def}} \lambda u.\lambda v.((u)\text{T})v$$

Church numerals

$0 =_{\mathrm{def}} \lambda f.\lambda x.x$

$1 =_{\mathrm{def}} \lambda f.\lambda x.(f)x$

$\mathsf{n} =_{\mathrm{def}} \lambda f.\lambda x.(f)(f)\ldots(f)x$
with n times f

$\mathsf{Succ} =_{\mathrm{def}} \lambda n.\lambda f.\lambda x.(f)((n)f)x$

$+ \equiv \lambda m.\lambda n.\lambda f.\lambda x.((m)f)((n)f)x$

$* \equiv \lambda m.\lambda n.\lambda f.(m)(n)f$

Overview

Frege's principle

The meaning of an expression is uniquely determined by the meanings of its parts and their mode of combination.

Motivation : Humbolt's view on finite means for infinite sentences

Consequences

- Locality principle : lexical items have a meaning that is independant of the expression they occur in.
- Substitution principle : synonymous expressions may be substituted for each other without changing the meaning of the complex expression in which they occur.
- Parts of well formed sentences have « meaning »
- Meanings can be « composed » : Frege's saturation idea

λ-terms can represent individual meanings and functional application can represent semantic composition.

Overview

Type theory

1. e is a type
2. t is a type
3. if a and b are types, then $\langle a, b \rangle$ is a type

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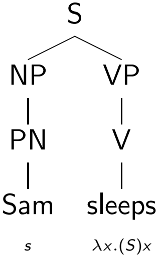
- $D_e = A$
- $D_t = \{0, 1\}$
- $D_{\langle a,b\rangle}$ = the set of mappings from D_a to D_b.

Meaningful expressions

For a, b types :

- variables and individual constants of type a belong to ME_a.
- if $\alpha \in ME_{\langle a,b \rangle}$ and $\beta \in ME_a$ then $(\alpha)\beta \in ME_b$.
- if u is a variable of type a and $\alpha \in ME_b$, then $\lambda u.\alpha \in ME_{\langle a,b \rangle}$.
- if φ and ψ are in ME_t, then the following expressions are also in ME_t : $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$.
- if φ is in ME_t and u is a type a variable, then $\forall u\varphi$ and $\exists u\varphi$ are in ME_t.

Overview



$$
\begin{array}{cccc}
S & \rightarrow & NP & VP \\
[\![S]\!] & \leftarrow & ([\![VP]\!]) & [\![NP]\!] \\
0 & \leftarrow & (2) & 1 \\
\hline
NP & \rightarrow & PN & \\
0 & \leftarrow & 1 & \\
\hline
VP & \rightarrow & V & \\
0 & \leftarrow & 1 & \\
\hline\hline
PN & \rightarrow & \mathrm{Sam} & \\
0 & \leftarrow & s & \\
\hline
V & \rightarrow & \mathrm{sleeps} & \\
0 & \leftarrow & \lambda x.(S)x &
\end{array}
$$

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Roadmap for the fragment

• A cat enters	$\exists x\ (Cx \wedge Ex)$
• Sam likes Pam	Lsp (or $((L)s)p$
• Everyone likes Pam	$\forall x\ (Px \rightarrow Lxp)$
• Everyone likes an actress	$\forall x\ (Px \rightarrow \exists y\ (Ay \wedge Lxy))$
• Sam is mortal	Ms
• Sam met a tall person	$\exists x((Px \wedge Tx) \wedge Msx)$
• Sam doesn't sleep	$\neg Ss$