A crash course in First Order Logic

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Propositional Logic

1. Base objects
   1.1 Propositions
   1.2 Logical connectives
2. Syntax
   2.1 wffs
   2.2 syntactic tree
3. Semantics
   3.1 Valuation
   3.2 Truth tables (simple and composite)
4. Reasoning
   4.1 Properties of formulae
   4.2 Relations between formulae
   4.3 Deduction theorem

Well-formed formulae

Let L_p be the language of propositional logic. The vocabulary of L_p comprises (i) a set of *proposition symbols* P, Q, R..., (ii) a unary connective $\neg$, (iii) binary connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and
(iv) parenthesis (&).
The **well formed formulae** (wffs) of L_p are given by:

(i). All proposition symbols are wffs.

(ii). If φ is a wff of L_p, then $\neg\varphi$ is also a wff of L_p.

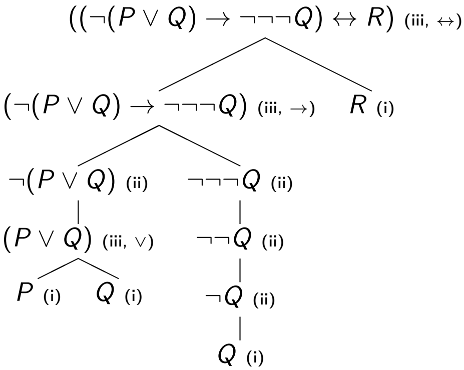
(iii). If φ and ψ are wffs of L_p, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.

(iv). Nothing else is a wff
(Nothing that cannot be constructed by successive steps of (i), (ii) or (iii) is a wff).

Well-formed formulae

$$
\begin{array}{lcl}
\text{WF} & \rightarrow & \text{P} \mid \text{Q} \mid \text{R} \\
\text{WF} & \rightarrow & (\ \text{WF BOP WF}\ ) \\
\text{WF} & \rightarrow & \neg\ \text{WF} \\
\text{BOP} & \rightarrow & \wedge \mid \vee \mid \rightarrow
\end{array}
$$

Syntactic tree



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Valuation

Let V be a *truth assignment* (or *valuation*) that maps all proposition symbols to a truth value (it can also be seen as a *model*). Then predicate calculus can be defined inductively as follows:

(i). If φ is a proposition symbol, then $[\![\varphi]\!]_V = V(\varphi)$;

(ii). If φ is a wff, then $[\![\neg\varphi]\!] = 1$ if and only if $[\![\varphi]\!] = 0$;

(iii). If φ and ψ are wffs, then

- $[\![(\varphi \wedge \psi)]\!] = 1$ iff $[\![\varphi]\!] = 1$ and $[\![\psi]\!] = 1$;
- $[\![(\varphi \vee \psi)]\!] = 0$ iff $[\![\varphi]\!] = 0$ and $[\![\psi]\!] = 0$;
- $[\![(\varphi \rightarrow \psi)]\!] = 0$ iff $[\![\varphi]\!] = 1$ and $[\![\psi]\!] = 0$;
- $[\![(\varphi \leftrightarrow \psi)]\!] = 1$ iff $[\![\varphi]\!] = [\![\psi]\!]$;

Truth tables

φ	$\neg\varphi$
0	1
1	0

φ	ψ	$\varphi \land \psi$
0	0	0
0	1	0
1	0	0
1	1	1

φ	ψ	$\varphi \lor \psi$
0	0	0
0	1	1
1	0	1
1	1	1

φ	ψ	$\varphi \to \psi$
0	0	1
0	1	1
1	0	0
1	1	1

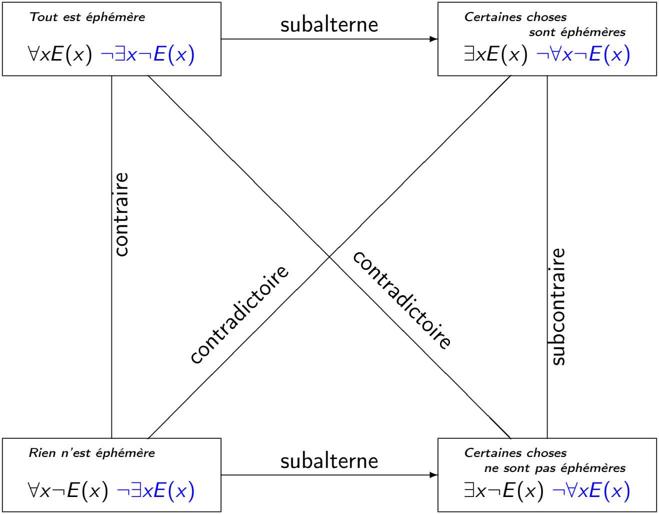
φ	ψ	$\varphi \leftrightarrow \psi$
0	0	1
0	1	0
1	0	0
1	1	1

Composite truth table

$((p$	$\wedge$	$(q$	$\rightarrow$	$r))$	$\vee$	$(r$	$\rightarrow$	$p))$
0	0	0	1	0	1	0	1	0
0	0	0	1	1	0	1	0	0
0	0	1	0	0	1	0	1	0
0	0	1	1	1	0	1	0	0
1	1	0	1	0	1	0	1	1
1	1	0	1	1	1	1	1	1
1	0	1	0	0	1	0	1	1
1	1	1	1	1	1	1	1	1

Predicate Logic

Quantifiers



Syntax I

Definition 1

(i) If A is a predicate constant, of arity n, and each $t_1 ... t_n$ an individual constant or variable, then $A(t_1, ..., t_n)$ is a wff.

(ii) If φ is a wff, then so is $\neg\varphi$.

(iii) If φ and ψ are wffs, then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, and $(\varphi \leftrightarrow \psi)$.

(iv) If φ is a wff and x a variable, then $\forall x \varphi$ and $\exists x \varphi$ are wffs.

(v) Nothing else is a wff.

Syntax II

Definition 2

If $\forall x \psi$ is a sub-formula of φ, then ψ is called the **scope** of this occurrence of the quantifier $\forall x$ in φ. Same definition for $\exists x$.

Definition 3

(a) An occurrence of a variable x in the formula ϕ (which is not part of a quantifer) is called **free** if this occurrence of x is not in the scope of a quantifier $\forall x$ ou $\exists x$ occurring in ϕ.

(b) If $\forall x \psi$ (or $\exists x \psi$) is a sub-formula of ϕ and x is free in ψ, then this occurrence of x is called **bound** by the quantifier $\forall x$ (or $\exists x$).

Definition 4

A **sentence** is a formula with no free variable.

Tarskian truth definition

Let $[\![\alpha]\!]^g_{\mathcal{M}}$ be the denotation of α in the model $\mathcal{M} = \langle D, I \rangle$ and with the assignment g.

$[\![t]\!]^g_{\mathcal{M}} = I(t)$ if t is an individual constant

$[\![t]\!]^g_{\mathcal{M}} = g(t)$ if t is a variable

$$[\![P(t_1, ...t_n)]\!]^g_{\mathcal{M}} = 1 \text{ iff } \langle [\![t_1]\!]^g_{\mathcal{M}}, ...[\![t_n]\!]^g_{\mathcal{M}} \rangle \in I(P).$$

If φ and ψ are wfss,

$[\![\neg\varphi]\!]^g_{\mathcal{M}} = 1$	iff	$[\![\varphi]\!]^g_{\mathcal{M}} = 0$	
$[\![(\varphi \wedge \psi)]\!]^g_{\mathcal{M}} = 1$	iff	$[\![\varphi]\!]^g_{\mathcal{M}} = 1$	and $[\![\psi]\!]^g_{\mathcal{M}} = 1$
$[\![(\varphi \vee \psi)]\!]^g_{\mathcal{M}} = 1$	iff	$[\![\varphi]\!]^g_{\mathcal{M}} = 1$	or $[\![\psi]\!]^g_{\mathcal{M}} = 1$
$[\![(\varphi \rightarrow \psi)]\!]^g_{\mathcal{M}} = 1$	iff	$[\![\varphi]\!]^g_{\mathcal{M}} = 0$	or $[\![\psi]\!]^g_{\mathcal{M}} = 1$

$$\boxed{[\![\exists y\ \varphi]\!]^g_{\mathcal{M}} = 1 \text{ iff there is a } d \in D \text{ s.t. } [\![\varphi]\!]^{g[y/d]}_{\mathcal{M}} = 1}$$

similarly,

$$\boxed{[\![\forall y\ \varphi]\!]^g_{\mathcal{M}} = 1 \text{ iff for all } d \in D,\ [\![\varphi]\!]^{g[y/d]}_{\mathcal{M}} = 1}$$

Il φ is a sentence:

$$\underline{[\![\varphi]\!]_{\mathcal{M}} = 1 \text{ iff there is an assignment } g \text{ such that } [\![\varphi]\!]^g_{\mathcal{M}} = 1}$$

Equivalences I

- Bound variables are "dummy": their name no longer matters.

$$\forall x\ Fx \quad \equiv \quad \forall y\ Fy$$

But beware of unintended captures:

$$\forall x\ (Fx \wedge Gy) \quad \not\equiv \quad \forall y\ (Fy \wedge Gy)$$

- Duality rules (*de Morgan* laws)

$$\forall x\ \alpha \quad \equiv \quad \neg\exists\ \neg\alpha$$

for instance:

$$\forall x\ Rx \quad \equiv \quad \neg\exists\ \neg Rx$$

All is relative $\approx$ *Nothing is absolute ($\approx$ non relative)*

$$\forall x\ (Px \to Kx) \quad \equiv \quad \neg\exists x\ (Px \wedge \neg Kx)$$

All professors are kind $\approx$ *There are no non-kind professors*

Other variants:

$$\exists x\ \alpha \quad \equiv \quad \neg\forall x\ \neg\alpha$$

$$\neg\exists x\ \alpha \quad \equiv \quad \forall x\ \neg\alpha$$

$$\neg\forall x\ \alpha \quad \equiv \quad \exists x\ \neg\alpha$$

Equivalences II

- Distribution rules:

$$\forall x\,(\alpha \land \beta) \quad \equiv \quad (\forall x\,\alpha \land \forall x\,\beta)$$

All is rare and expensive $\approx$ *All is rare and all is expensive*

But:

$$\forall x\,(\alpha \lor \beta) \quad \not\equiv \quad (\forall x\,\alpha \lor \forall x\,\beta)$$

All is either relative or absolute $\not\approx$ *Either all is relative or all is absolute*

$$\exists x\,(\alpha \lor \beta) \quad \equiv \quad (\exists x\,\alpha \lor \exists x\,\beta)$$

But:

$$\exists x\,(\alpha \land \beta) \quad \not\equiv \quad (\exists x\,\alpha \land \exists x\,\beta)$$

$$\exists x\,(\alpha \rightarrow \beta) \quad \equiv \quad (\forall x\,\alpha \rightarrow \exists x\,\beta)$$

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Equivalences III

- Conditional distribution ($\bar{\beta}$ doesn't contain free occurrences of x)

$$
\begin{aligned}
\bar{\beta} &\equiv \forall x \bar{\beta} \\
\bar{\beta} &\equiv \exists x \bar{\beta} \\
\\
\forall x\ (\alpha \vee \bar{\beta}) &\equiv (\forall x\ \alpha \vee \bar{\beta}) \\
\exists x\ (\alpha \wedge \bar{\beta}) &\equiv \exists x\ \alpha \wedge \bar{\beta} \\
\forall x\ (\alpha \rightarrow \bar{\beta}) &\equiv \exists x\ \alpha \rightarrow \bar{\beta}
\end{aligned}
$$

Every entity is such that if it breaks, there is noise ≈ *If some entity breaks,*

$$\forall x\ (\bar{\beta} \rightarrow \alpha) \equiv \bar{\beta} \rightarrow \forall x\ \alpha$$

For all person, if there is noise, s/he is upset ≈ *If there is noise, every*