Predicate Logic

N° 1.
Translate as precisely as possible the following sentences into predicate logic. Explain the interpretation of non logical constants when it is not obvious. In case of ambiguity, propose as many formulae as necessary.

(1)
a. All architects have built a bridge.
b. Sam is depressed when no one understands her.
c. A child is confident only if no adult lies to him.
d. There are only two solutions to every problem.
e. Every student who solves a problem will explain it.
f. Any company which disappoints some customers will disappear if they all complaint.
g. Either no student will find an answer or all of them will.
h. ~~A doctor will be sent to anyone who refuses that someone loves her.~~
A doctor will be sent to anyone who rejects someone who loves her.

N° 2. Among the formulae given in (2), indicate those which are appropriate representations for (3). Provide an explanation for those which are not appropriate.

(2)
a. $\forall x \ \Big(\big(Px \land \exists y \ (Cy \land Rxy)\big) \to Sxy\Big)$
b. $\forall x \forall y \ \Big((Px \land Cy \land Rxy) \to Sxy\Big)$
c. $\forall x \ \Big(Px \to \forall y \ \big((Cy \land Rxy) \to Sxy\big)\Big)$
d. $\forall x \exists y \ \Big((Px \land Cy \land Rxy) \to Sxy\Big)$

(3) Any person who raises a child is sacrificing for them.
Convention: $Sxy = x$ is sacrificing for y.

N° 3. Let us consider the following syllogism:

(4)
If a student fails, they didn't work correctly
Alex is a student
Alex failed
$\therefore$ Alex didn't work correctly

1. Translate this syllogism into predicate logic.
2. Give the simplest possible model in which all the propositions are true.
3. A model-theoretic proof for this syllogism would try to prove that every model that makes the premisses true would also satisfy the conclusion. Sketch informally a proof of this sort.

Answer n° 1.

Translate as precisely as possible the following sentences into predicate logic. Explain the interpretation of non logical constants when it is not obvious. In case of ambiguity, propose as many formulae as necessary.

General remark: every formula has to have a clear, non ambiguous syntactic structure. This can be achieved via different means (brackets, priorities of connectives, notation conventions...). What I proposed in class was to have one single rule: *binary connectives (and only them) each come along with a pair of brackets.* I also used an additional convention that with associative connectives, brackets whose removal do not introduce a semantic ambiguity (while introducing by construction a syntactic ambiguity) can be dispensed of. The typical example is (5a) which is syntactically ambiguous between the two structures in (5b), this ambiguity being harmless because of the associativity of $\wedge$.

Another typical case, much more problematic, is illustrated by (5c): we didn't offer in class any way to disambiguate such an ill-formed formula, whose ambiguity is not harmless (5d). There are at least two ways to still provide a syntactic analysis for (5c) (without explicitely adding brackets): one is to assume a commonly adopted convention that external brackets are omitted (in other words, if a pair of brackets is missing, assume that the whole formula is within brackets). This option leads to the struture on the right in (5d) (by the way, a formula where the second occurrence of x is free). The other possible option is to assume that $\rightarrow$ has priority over $\forall$. This would lead to the structure of the left formula in (5d).

(5) a. $(a \wedge b \wedge c)$
b. $(a \wedge (b \wedge c))$ *vs.* $((a \wedge b) \wedge c)$
c. $\forall x\ Ax \rightarrow Bx$
d. $\forall x\ (Ax \rightarrow Bx)$ *vs.* $(\forall x\ Ax \rightarrow Bx)$

My advice: to ensure that your answer is not syntactically ambiguous, insert brackets as required or make explicit what convention you want to be at play.

(1a) *All architects have built a bridge.*
The formula that is closest to the surface structure of the sentence is (6a) (with $Cxy = x$ *built* y), which can also be written (6b) if you insist in having a prenex form. It is also imaginable that *a bridge* take a large scope, however implausible it may be ("*there is a bridge that all architects have built*"). In that case a possible formula is (6c). Also note that (6d) could be proposed with $Bx = x$ *built a bridge*. But it is not entirely satisfactory since the predicate can (and should) be further decomposed to account for the (somewhat implicit) quantification at stake.

(6) a. $\forall x\ (Ax \rightarrow \exists y\ (By \wedge Cxy))$
b. $\forall x \exists y\ (Ax \rightarrow (By \wedge Cxy))$
c. $\exists y\ (By \wedge \forall x\ (Ax \rightarrow Cxy))$
d. $\forall x\ (Ax \rightarrow Bx)$

(1b) *Sam is depressed when no one understands her.*
The pseudo-sentence *Sam is depressed when* φ would correspond to the formula (7a). The sentence *No one understands Sam* can be translated as (7b) or (7c). Putting the two together gives (7d). Note that (7e) is syntactically ambiguous unless we add a convention that only external brackets can be omitted (in which

case it is equivalent to (7d)). Also note that (7f) means that Sam is depressed as soon as any person doesn't understand her. This is too strong and does not correspond to the initial sentence.

(7) a. $(\varphi \to Ds)$
b. $\neg\exists x\ (Px \wedge Uxs)$
c. $\forall x\ (Px \to \neg Uxs)$
d. $(\neg\exists x\ (Px \wedge Uxs) \to Ds)$
e. $\neg\exists x\ (Px \wedge Uxs) \to Ds$
f. $\forall x\ \big((Px \wedge \neg Uxs) \to Ds\big)$

(1c) *A child is confident only if no adult lies to him.*
A pseudo-sentence "φ only if ψ" has the logical structure (8a). *A child* is interpreted universally here (*any child*), so we have a global structure like (8b), where ξ corresponds to "if x is confident then no adult lies to x". The final proposition is (8c) ($Tx = x$ is confident), of which (8d) and (8e) are possible variants among many. An alternative reading (not very accessible) would interpret *a child* existentially ("*there is child such that he is confident...*"). (8f) is a possible representation for this reading.

(8) a. $(\varphi \to \psi)$
b. $\forall x\ (Cx \to \xi)$
c. $\forall x\ \Big(Cx \to \big(Tx \to \neg\exists y\ (Ay \wedge Lyx)\big)\Big)$
d. $\forall x\ ((Cx \wedge Tx) \to \neg\exists y\ (Ay \wedge Lyx)))$
e. $\forall x\ (Cx \to (Tx \to \forall y\ (Ay \to \neg Lyx)))$
f. $\exists x\ (Cx \wedge (Tx \to \forall y\ (Ay \to \neg Lyx)))$

(1d) *There are only two solutions to every problem.*
We can assume that this sentence carries two distinct propositions (9a) and (9b). It is usually considered that (9a) is presupposed, while (9b) is asserted, and we can adopt here a so-called Russelian representation by simply conjoining the two contents. Let's also assume that if x is a problem, a « *solution to x* » is an entity that solves x, to avoid the problem we may face by stating something like *x is a solution that doesn't solve the problem y*. Let's finally assume that the two solutions are different. We need to introduce in the FOL language the *equal* sign, without which it's not possible to talk about identity or difference. A possible representation for (9a) is (9c). (Note that $y \neq z$ is just a convenient notation for $\neg\ y = z$ — no brackets here, given our notation conventions.) The version without the inequality (9d) would be true even when there is only one solution, since nothing prevents the two variables y and z to refer to the same entity.
A possible representation for (9b) could be (9e) ("*it is not the case that there are three different solutions to every problem*"). So a possible formula for the initial sentence could be the conjunction of (9c) and (9e). However, a more compact option is usually taken: we can chose to express in the same formula that there are two solutions, and that everything different from those two solutions cannot be a solution (9f). Note that in (9c-f) we refrained from inserting unnecessary brackets in sequences of conjunctions. Note also that there are two equivalent ways to express the "restriction" part of the content (9g).

(9) a. *There are (at least) two solutions to every problem*
b. *There are not more than two solutions to every problem*

c. $\forall x\ (Px \to \exists y \exists z\ ((Syx \wedge Szx) \wedge y \neq z))$

d. $\forall x\ (Px \to \exists y \exists z\ (Syx \wedge Szx))$

e. $\neg\forall x\ (Px \to \exists u \exists v \exists w\ (Sux \wedge Svx \wedge Swx \wedge u \neq v \wedge u \neq w \wedge v \neq w))$

f. $\forall x\ \Big(Px \to \exists y \exists z\ \big(y \neq z \wedge Syx \wedge Szx \wedge \forall u\ ((u \neq y \wedge u \neq z) \to \neg Sux)\big)\Big)$

g. $\forall u\ \ ((u \neq y \wedge u \neq z) \to \neg Sux) \quad \Leftrightarrow \quad \forall u\ \ (Sux \to (u = y \vee u = z))$

(1e) *Every student who solves a problem will explain it.*
This a typical donkey sentence. If *a problem* is taken existentially, as seems natural, we have an interpretation that could be paraphrased as (10a). A strictly compositional treatment of this paraphrase will yield (10b) which is well-formed but unfortunately not closed, which means that it cannot receive a truth value, and therefore cannot serve as a representation for the initial sentence (which can receive a truth value). Instead we can offer (10c), a less compositional but more accurate representation for the initial sentence. A more legible (logically equivalent) variant is (10d). An alternative reading would have "a problem" have the largest possible scope ("*there is a problem which is such that every student...*"). The formula (10e) is a possible representation for this reading.

(10) a. *For every student, if there is a problem that they solve, then they will explain it*

b. $\forall x\ (Sx \to (\exists y\ (Py \wedge Sxy) \to Exy))$

c. $\forall x \forall y\ \Big(Sx \to \big((Py \wedge Sxy) \to Exy\big)\Big)$

d. $\forall x \forall y\ ((Sx \wedge Py \wedge Sxy) \to Exy)$

e. $\exists y\ \ (Py \wedge \forall x\ ((Sx \wedge Sxy) \to Exy))$

(1f) *Any company which disappoints some customers will disappear if they all complaint.*
The preferred reading of this sentence could be the paraphrase (11a), but an alternative reading would be paraphrased by (11b). In this context, *customer* should probably be taken to mean "*customer of the company*": a binary relation rather than a unary property. To express the alternative reading, it's enough to remove the underlined part of the formula (11b) (and brackets accordingly).

(11) a. *For every company, if there are customers that it disappoints, then if all the disappointed customers complaint, it will disappear.*

b. *For every company, if there are customers that it disappoints, then if all the ~~disappointed~~ customers complaint, it will disappear.*

c. $\forall x\ \Big(Cx \to \Big(\exists y(Cyx \wedge Dxy) \to \big(\forall z\ \big((Czx \underline{\wedge Dxz}) \to Pz\big) \to Dx\big)\Big)\Big)$

(1g) *Either no student will find an answer or all of them will.*
This a disjunction of two independant propositions. The noun *answer* could be taken as expressing a binary relation (an answer to a specific question). Since the sentence does not refer to a question, we make the simplifying assumption that *finding an answer* is interpreted as *there exists an answer that one finds*. The equivalent variant (12b) makes more visible the parallellism between the two disjuncts. A wide scope reading of *an answer* ("*there is an answer such that either... or...*") doesn't seem very accessible (12c).

(12) a. $\Big(\neg\exists x\ (Sx \wedge \exists y\ (Ay \wedge Fxy)) \ \ \vee\ \ \forall x\ (Sx \to \exists y\ (Ay \wedge Fxy))\Big)$

b. $(\forall x\ (Sx \to \neg\exists y\ (Ay \wedge Fxy)) \ \ \vee\ \ \forall x\ (Sx \to \exists y\ (Ay \wedge Fxy)))$

c. $\exists y\ \ \big(Ay \wedge \big(\forall x(Sx \to \neg Fxy) \vee \forall x(Sx \to Fxy)\big)\big)$

(1h) ~~*A doctor will be sent to anyone who refuses that someone loves her.*~~
This sentence was given by mistake since it comprises a part that is not representable in first order logic: the relation denoted by *refuse* whose second argument is not an entity but a proposition. A not very natural way to solve the problem would have been to define a first order predicate R such that Rxy means x refuses that someone loves y. A possible formula would be (13a), where *a doctor* is taken to have wide scope over the rest (which may not be the most natural reading) and $Suv = u$ is sent to v.

A doctor will be sent to anyone who rejects someone who loves her.
This alternative sentence with no second order predicate was given in class. Let's consider first the pseudo-sentence (13b). We have here the familiar ambiguity depending on the scope interaction between the two quantified NPs. A "congruent" reading would be (13c), while an "inverse scope" reading would be (13d). Let's now unfold the predicate φ. Consider (13e) as an intermediary step. We assume that *her* is referring back to the subject (otherwise we have a non resolved anaphora). We have again two readings depending on how *someone* is interpreted. A wide scope reading (*there is someone who loves Jim that she rejects*) would be represented as (13f), while a narrow scope version would be (13g), in which *someone* is interpreted as *anyone*. So finally, there are at least four different ways to build a representation, given the ambiguities we found. My preferred reading is (13h).

(13) a. $\exists y\,(Dy \wedge \forall x\,((Px \wedge Rxx) \rightarrow Syx))$
b. *A doctor will be sent to anyone who* φ
c. $\exists y\,(Dy \wedge \forall x\,((Px \wedge \varphi(x)) \rightarrow Syx))$
d. $\forall x\,((Px \wedge \varphi(x)) \rightarrow \exists y\,(Dy \wedge Syx))$
e. *Jim rejects someone who loves her*
f. $\exists z\,((Pz \wedge Lzj) \wedge Rjz)$
g. $\forall z\,((Pz \wedge Lzj) \rightarrow Rjz)$
h. $\forall x\,\Big(\big(Px \wedge \forall z\,((Pz \wedge Lzx) \rightarrow Rxz)\big) \rightarrow \exists y\,(Dy \wedge Syx)\Big)$

Answer n° 2. *Among the formulae given in (15), indicate those which are appropriate representations for (14). Provide an explanation for those which are not appropriate.*

(14) *Any person who raises a child is sacrificing for them.*

(15) a. $\forall x\,\Big(\big(Px \wedge \exists y\,(Cy \wedge Rxy)\big) \rightarrow Sxy\Big)$
The last occurrence of y is free. It is therefore **not appropriate**.
b. $\forall x \forall y\,\Big((Px \wedge Cy \wedge Rxy) \rightarrow Sxy\Big)$
This formula is **appropriate**
c. $\forall x\,\Big(Px \rightarrow \forall y\,\big((Cy \wedge Rxy) \rightarrow Sxy\big)\Big)$
This formula is **appropriate** (and logically equivalent to the previous one)
d. $\forall x \exists y\,\Big((Px \wedge Cy \wedge Rxy) \rightarrow Sxy\Big)$
This formula is **not appropriate**, since it states that for every person there is an entity for which it is true that **if it is a child raised by this person** then the person will sacrifice for it. This will be trivially true for entities that are not children (for example), and the formula will consequently be true when (14) is false.

Answer n° 3. *Let us consider the following syllogism:*

(16)

	If a student fails, they didn't work correctly
	Alex is a student
	Alex failed
∴	*Alex didn't work correctly*

1. *Translate this syllogism into predicate logic.*

	$\forall x\ ((Sx \wedge Fx) \rightarrow \neg Wx)$
	Sa
	Fa
∴	$\neg Wa$

2. *Give the simplest possible model in which all the propositions are true.*

   The simplest model contains one entity (the denotation of *Alex*), let us call it d: $d \in \mathcal{U}$ and $[\![a]\!] = d$). Then if $d \in [\![S]\!]$,
   $d \in [\![F]\!]$, and
   $d \notin [\![W]\!]$
   then the propositions are all true, including the first premise.

3. *A model-theoretic proof for this syllogism would try to prove that every model that makes the premisses true would also satisfy the conclusion. Sketch informally a proof of this sort.*

   Any model that makes the first premise true would be such that the intersection between $[\![S]\!] \cap [\![F]\!]$ and $[\![W]\!]$ is empty, while there would have to be at least one entity in $[\![S]\!] \cap [\![F]\!]$, the denotation of a, to make the two other premises true. Since $[\![W]\!]$ cannot contain any member of $[\![S]\!] \cap [\![F]\!]$ it will never contain $[\![a]\!]$. Therefore Wa will always be false.